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ON THE CONSTRUCTION OF A STABLE BRIDGE IN A RETENTION GAME*

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A construction procedure is described for a u-stable bridge /1/ in a defferential retention game. Concrete classes of games reducing to a retention game are examined. Examples are presented.

1. Consider a controlled process whose equations of motion are

$$z^{\cdot} = f(t, z, u, v), \quad z \in R^n, \quad u \in U(t), \quad v(t) \in V(t) \tag{1.1}$$

A segment I of the real line is prescribed. For each $t \in I$ the sets $U(t)$ and $V(t)$ are compacta in R^n and depend measurably on t on segment I. A family of sets $W(t) \subset R^n$ satisfying the closure condition

$$t_i \to t, \ x_i \to x, \ x_i \in W(t_i) \Rightarrow x \in W(t) \tag{1.2}$$

is prescribed on segment I. An initial position $t_0 \in I$, $z(t_0) \in W(t_0)$ and a number $p > t_0$, $p \in I$, are specified. The first player's purpose is to retain, by choosing a control u, the point $z(t)$ in set $W(t)$ for all $t_0 \leqslant t \leqslant p$ for any behavior of the second player. We make the following assumptions regarding the right-hand side of system (1.1): a) for any initial condition $t_1 \in I, z(t_1) \in R^n$ and any controls $u(t) \in V(t)$, $v(t) \in V(t)$ measurable on segment I the system (1.1) has a unique solution defined on segment I, b) for a control $v(t) \in V(t)$ measurable on I, from every infinite sequence of solutions $z_i(t)$ of system (1.1) with controls $u_i(t) \in U(t)$ and initial conditions $t_i \to t^0$, $z(t_i) \to z^0$ we can pick out a sequence uniformly convergent on I, where the limit function is a solution with the same control $v(t)$ and with some measurable control $u(t) \in U(t)$.

To construct the u-stable bridge /1/ corresponding to the problem being examined we use the multivalued mapping introduced in /2/ for stationary games. Let a set $X \subset R^n$ and a number $t_1 \leqslant \tau$ be specified. Then $T_{t_1}^{\tau}(X)$ is the set of points $z \in R^n$ for each of which we can find, for any control $v(t) \in V(t)$ measurable on $[t_1, \tau]$, a control $u(t) \in U(t)$ measurable on this interval, such that $z(\tau) \in X$. Here $z(\tau)$ is the value of the solution of system (1.1) with initial conditions $z(t_1) = z$. Under the assumptions made the mapping T has the following properties:

1) if set X is closed, $x_i \to x$, $t_i \to t$, $x_i \in T_{t_i}^{\tau}(X)$, then $x \in T_t^{\tau}(X)$;

2) if sets X_i are closed and $X_{i+1} \subset X_i$, then

$$\bigcap_{i \geqslant 1} T_t^{\tau}(X_i) = T_t^{\tau}\Big(\bigcap_{i \geqslant 1} X_i\Big);$$

3) if $X \subset X_1$, then $T_t^{\tau}(X) \subset T_t^{\tau}(X_1)$;

4) $T_t^t(X) = X$;

5) the inclusion $T_t^{t_1}(T_{t_1}^{\tau}(X)) \subset T_t^{\tau}(X)$ is fulfilled for any $t \leqslant t_1 \leqslant \tau$ and any set $X \subset R^n$.

For $t \leqslant p$ we define a family of sets $W^k(t)$ by the recurrence relation

$$W^0(t) = W(t), \ \ldots, \ W^k(t) = \bigcap_{t \leqslant \tau \leqslant p} T_t^{\tau}(W^{k-1}(\tau)) \tag{1.3}$$

The next properties follow from the closure condition (1.2) and the properties of mapping T: 1) the set $W^k(t)$ satisfies the closure condition (1.2); 2) $W^{k+1}(t) \subset W^k(t)$; 3) $W^k(p) = W(p)$.

Lemma 1. Let the initial conditions be such that $z(t_0) \equiv W^k(t_0)$ for some $k \geqslant 1$. Then a second player's ε-strategy /2/ exists leading the trajectory $z(t)$ out of the family of sets $W(t)$ by the instant p.

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Proof. From the lemma's hypothesis and from relations (1.3) it follows that $z(t_0) \in T_{t_0}^{\tau}(W^{k-1}(\tau))$ for some $t_0 \leqslant \tau \leqslant p$. Therefore, the second player can construct a control measurable on interval $[t_0, \tau]$ such that $z(\tau) \in W^{k-1}(\tau)$ for any measurable control of the first player. If $\tau = p$, then $z(p) \in W(p)$. We need to carry out this argument k times. It can be shown the second player's control construction rule presented is realized by a certain ε-strategy whose rigorous formalization is contained in /3/.

For each $t \leqslant p$ we set

$$M(t) = \bigcap_{k \geqslant 1} W^k(t) \tag{1.4}$$

Then, as follows from the properties of $W^k(t)$, the set $M(t)$ satisfies the closure condition (1.2) for $t \leqslant p$.

Lemma 2. $T_t^{\tau}(M(\tau)) \supset M(t)$ for $t \leqslant \tau \leqslant p$.

Proof. From relations (1.3) and (1.4), the property 2) of mapping T, and the properties of sets $W^k(t)$ it follows that

$$M(t) \subset \bigcap_{k \geqslant 1} T_t^{\tau}(W^{k-1}(\tau)) \quad T_t^{\tau}(M(\tau)).$$

From the lemmas proved it follows that the family of sets (1.4) is a maximal u-stable bridge in the problem of retention up to instant p.

Corollary. Let a number $k \geqslant 0$ exist such that $W^k(t) \subset W^{k+1}(t)$ for $t \leqslant p$. Then $M(t)$ $W^k(t)$.

Example. Consider the one-type game with simple motion

$$z^{\cdot} = -u + v, \quad u \in \alpha(t) S, \quad v \in \beta(t) S$$

Here S is a convex compactum in R^n containing the origin, $\alpha(t) \geqslant 0$ and $\beta(t) \geqslant 0$ are functions summable on segment I. Then, using the definition of geometric difference $\overset{*}{-}$ /4/, we have

$$T_t^{\tau}(X) = \left(X + \int_t^{\tau} \alpha(r)\, dr S\right) \overset{*}{-} \int_t^{\tau} \beta(r)\, dr\, S \tag{1.5}$$

Let $W(t) = \delta(t) S$, where $\delta(t) \geqslant 0$ is a function continuous on I. We define the number

$$b = \inf\left\{t \in I : \delta(\tau) \geqslant \int_t^{\tau} (\beta(r) - \alpha(r))\, dr, \quad t \leqslant \tau \leqslant p\right\} \tag{1.6}$$

Then from (1.5) we can obtain that

$$T_t^{\tau}(W(\tau)) = \left(\delta(\tau) + \int_t^{\tau} (\alpha(r) - \beta(r))\, dr\right) S$$

for $b \leqslant t \leqslant \tau \leqslant p$. Consequently,

$$W^1(t) = \delta_1(t) S, \quad \delta_1(t) = \min_{t \leqslant \tau \leqslant p} \left(\delta(\tau) + \int_t^{\tau} (\alpha(r) - \beta(r))\, dr\right) \tag{1.7}$$

for $b \leqslant t \leqslant p$. If $t < b$, then a number $t < \tau \leqslant p$ exists for which the set $T_t^{\tau}(W(\tau))$ is empty. Therefore, the set $W^1(t)$ is empty for $t < b$. By analogous arguments we can prove the equality $W^2(t) = W^1(t)$ for $t \in I$.

Let us prove one further property of sets (1.3) and (1.4), to be used subsequently. Let a sequence of families of sets $W_i(t)$ $(t \in I)$ satisfying the closure condition (1.2) be specified. We set

$$W_0(t) = \bigcap_{i \geqslant 1} W_i(t)$$

For $W_i(t)$ we construct sets $W_i^k(t)$ and $M_i(t)$ by formulas (1.3) and (1.4) for each $i = 0, 1, \ldots$

Lemma 3. Let $W_{i+1}(t) \subset W_i(t)$ for $t \leqslant p$ and for all $i \geqslant 1$. Then

$$\bigcap_{i \geqslant 1} W_i^k(t) = W_0^k(t), \quad \bigcap_{i \geqslant 1} M_i(t) = M_0(t) \tag{1.8}$$

for $t \leqslant p$.

Proof. At first we show that $W_{i+1}^k(t) \subset W_i^k(t)$. This inclusion is fulfilled when $k=0$. Suppose that it is fulfilled for k for all $t \in I, t \leqslant p$. Then

$$W_{i+1}^{k+1}(t) = \bigcap_{t \leqslant \tau \leqslant p} T_t^\tau (W_{i+1}^k(\tau)) \subset \bigcap_{t \leqslant \tau \leqslant p} T_t^\tau (W_i^k(\tau)) = W_i^{k+1}(t)$$

By induction on k we prove the first equality in (1.8). It is fulfilled when $k=0$. Suppose that it is fulfilled for k. Then from the inclusion proved and from property 3) of mapping T follows

$$\bigcap_{i \geqslant 1} W_i^{k+1}(t) = \bigcap_{i \geqslant 1} \bigcap_{t \leqslant \tau \leqslant p} T_t^\tau (W_i^k(\tau)) = \bigcap_{t \leqslant \tau \leqslant p} T_t^\tau (\bigcap_{i \geqslant 1} W_i^k(\tau)) = W_0^{k+1}(t)$$

From the proved first equality in (1.8) it follows that

$$M_0(t) = \bigcap_{k \geqslant 1} W_0^k(t) = \bigcap_{k \geqslant 1} \bigcap_{i \geqslant 1} W_i^k(t) = \bigcap_{k \geqslant 1} \bigcap_{i \geqslant 1} W_i^k(t) = \bigcap_{i \geqslant 1} M_i(t)$$

2. Let us consider the following game: a closed set $Z \subset R^n$, a continuous function $g: Z \times I \to R$ bounded from below by number γ, and an initial position $t_0 \in I$, $z_0 \in R^n$ are prescribed. The first player's purpose is to retain the point $z(t)$ in set Z up to intstant p and to minimize the quantity

$$\max_{t_0 \leqslant t \leqslant p} g(z(t), t) \tag{2.1}$$

For each $\nu \geqslant \gamma$ we define the family of sets

$$W_\nu(t) = \{z \in Z: g(z, t) \leqslant \nu\}$$

on segment I. Then for $t_0 \leqslant t \leqslant p$ the inclusion $z(t) \in W_\nu(t)$ is equivalent to the requirement that the quantity (2.1) not exceed ν. For each $\nu \geqslant \gamma$ we construct the stable bridge $M_\nu(t)$ of (1.4). By $\nu_0 = \nu(z_0, t_0)$ we denote the lower bound of all numbers $\nu \geqslant \gamma$ for which

$$z_0 \in M_\nu(t_0) \tag{2.2}$$

From Lemma 3 it follows that inclusion (2.2) is fulfilled for $\nu = \nu_0$. Hence it follows that the first player can make the value of quantity (2.1) no larger than ν_0. We take $\nu < \nu_0$. Then inclusion (2.2) is not fulfilled. Therefore, the second player can lead point $z(t)$ out of set $W_\nu(t)$ by the instant p, i.e, make the value of quantity (2.1) larger than ν, or lead the point $z(t)$ out of set Z.

Note. We can use sets (1.3) for finding the value $\nu(z_0, t_0)$ in the game being considered. The numbers $\nu_k(z_0, t_0)$ are determined analogously. The sequence of these numbers grows and in the limit yields the game's value. Such sequential procedures for constructing the game's value were examined, for example, in /5—7/.

Example. Consider the example from section 1. We define the set $Z = \bigcup (\nu S)$, $\nu \geqslant 0$. We set

$$g(z) = \min\{\nu \geqslant 0: z \in \nu S\}$$

Then $W_\nu(t) = \nu S$. Therefore, for each $\nu \geqslant 0$, setting $\delta(\tau) = \nu$ in formulas (1.6) and (1.7), we obtain $b = b(\nu)$, $M_\nu(t) = \nu S$ for $b(\nu) \leqslant t \leqslant p$, and set $M_\nu(t)$ is empty for $t < b(\nu)$. Hence it follows that the game's value ν_0 for the initial position z_0, t_0 is determined as the least of the numbers $\nu \geqslant 0$ for which $b(\nu) \leqslant t_0$ and $z_0 \in \nu S$.

3. Consider the stationary retention game

$$z^\cdot = f(z, u, v), \quad u \in U, \quad v \in V$$

In this case $T_t^\tau(X) = T_{\tau - t}(X)$, where $T_\sigma(X)$ is the set of those points z for each of which we can find, for any measurable control $v(t) \in V$, a measurable control $u(t) \in U$ such that $z(\sigma) \in X$. Here $z(\sigma)$ is the value of the solution of system (2.1) with initial condition $z(0) = z$. Formulas (1.3) take the form

$$W^k(t) = \bigcap_{0 \leqslant \tau \leqslant p-t} T_\tau(W^{k-1}(t+\tau)) \tag{3.1}$$

In particular, if set $W(t) = Z$ is constant, then

$$W^1(t) = \bigcap_{0 \leqslant \tau \leqslant p-t} T_\tau(Z) \tag{3.2}$$

$$W^2(t) = \bigcap_{0 \leqslant r \leqslant p-t} T_r(\bigcap_{0 \leqslant \tau \leqslant p-t-r} T_\tau(Z)) \tag{3.3}$$

We introduce the multivalued mapping

$$L_\sigma(X) = \bigcap_{0\leqslant\tau\leqslant\sigma} T_\tau(X) \tag{3.4}$$

Theorem. If $L_r(L_\sigma(Z)) \supset L_{r+\sigma}(Z)$ for all $0 \leqslant r \leqslant p$, $0 \leqslant \sigma \leqslant p$, then $W^2(t) = W^1(t)$ for $0 \leqslant t \leqslant p$.

Proof. It is enough to show that $W^2(t) \supset W^1(t)$. From equalities (3.2)–(3.4) and the theorem's condition it follows that

$$W^2(t) \supset \bigcap_{0\leqslant r\leqslant p-t} L_r(L_{p-t-r}(Z)) \supset L_{p-t}(Z) = W^1(t)$$

Example. Consider the game with simple motion $z^{\cdot} = -u + v$, $u \in U$, $v \in V$. Here U and V are convex compacta in R^n. In this case /2/

$$L_\sigma(X) = \bigcap_{0\leqslant\tau\leqslant 1} ((X + \tau\sigma U) \stackrel{*}{-} \tau\sigma V) \tag{3.5}$$

Let us show that if Z is a convex set, then the condition of the preceding theorem is fulfilled. First of all, we note that if set X is convex, then so is set (3.5). In addition, it can be shown that

$$L_\sigma(X_1 + X_2) \supset L_\sigma(X_1) + X_2,\ L_\sigma(\sigma X) = \sigma L_1(X) \tag{3.6}$$

We take positive numbers r and σ. We set $Y = (r+\sigma)^{-1} Z$, $Z = \sigma Y + rY$. Then

$$L_\sigma(Z) \supset \delta L_1(Y) + rY$$

$$L_r(L_\sigma(Y)) \supset \sigma L_1(Y) + rL_1(Y) = (\sigma + r) L_1(Y) = L_{\sigma+r}((\sigma + r) Y) = L_{\sigma+r}(Z)$$

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